# 3/EH-29 (iii) (Syllabus-2015)

## 2018

( October )

## MATHEMATICS

( Elective/Honours )

( GHS-31 )

**( Algebra—II and Calculus—II )**

*Marks* : 75

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. (a) Show that the set $G = \{1, 2, 3, \cdots, p-1\}$ is a group of order $p-1$, the composition being ordinary multiplication modulo $p$, $p$ being a prime integer. 6

   (b) If $G$ is a finite group and $a \in G$, then prove that *(i)* if $a^m = e$, then $O(a)$ divides $m$ and *(ii)* $O(a) = O(a^{-1})$, where $e$ is the identity of $G$, $O(a)$ is the order of element $a$. 3+2=5

(c) For a given element $a$ in a group $G$, prove that the set

$$N(a) = \{x \in G \mid xa = ax\}$$

is a subgroup of a group $G$. 4

2. (a) Prove that every subgroup of a cyclic group is cyclic. 4

(b) Show that the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions in a group G, where $a, b \in G$. 5

(c) If an element $a$ of a group $G$ satisfies $a^2 = a$, then show that $a = e$. 2

(d) If $G = \langle \mathbb{Z}, + \rangle$ be the group of integer under ordinary addition and if $H = n\mathbb{Z}$, then find all the right cosets of $H$ in $G$, where $n$ is a fixed positive integer. 4

UNIT—II

3. (a) Find the range of the values of $k$ for which the roots of the equation

$$x^4 + 4x^3 - 8x^2 + k = 0$$

are all real. 5

(b) Find the polynomial $f(x+2)$, when

$$f(x) = 4x^5 + 6x^4 - 3x^3 + 5x - 2$$ 5



( Continued )



(c) Solve the equation

$$4x^4 + 8x^3 + 13x^2 + 2x + 3 = 0$$

given that sum of two of the roots is zero. 5

4. (a) Solve the equation

$$x^3 - 15x - 126 = 0$$

by Cardan's method. 5

(b) Use De Moivre's theorem to solve the equation

$$x^7 - 1 = 0$$ 5

(c) If $\alpha, \beta, \gamma$ be the roots of the cubic equation

$$x^3 + px^2 + qx + r = 0$$

find the value of $\sum \alpha^3\beta^3$. 5

UNIT—III

5. (a) Prove that a convergent sequence is bounded. Is the converse true? Justify your answer with an example. 3+1=4

(b) State Cauchy's general principle of convergence of a sequence and apply it to show that the sequence $\{x_n\}$ is divergent, if

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$ 2+3=5

( Turn Over )

(c) Show that the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}}$$

converges. Find $\lim_{n \to \infty} a_n$. 3+3=6

6. (a) Examine the convergence of the following series (any *two*) : 3×2=6

(i) $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n} \cdot \frac{1}{n}$

(ii) $2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \cdots + \frac{(n+1)x^n}{n^3} + \cdots$

(iii) $\sum_{n=1}^{\infty} \left( \sqrt{n^3+1} - \sqrt{n^3} \right)$

(b) What is an absolute convergent series? Test the absolute convergence of the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{(n+1)!}$$

2+3=5

(c) Determine the region of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 3^n}$$

4





## UNIT—IV

7. (a) State Rolle's theorem and give its geometrical interpretation. 2+2=4

(b) Show that the curve $y^3 = 8x^2$ is concave to the foot of the ordinate everywhere except at the origin. 3

(c) Verify Lagrange's mean value theorem for the function

$$f(x) = x(x-1)(x-2)$$

in $\left[0, \frac{1}{2}\right]$. 4

(d) Find the asymptotes of

$$xy^2 - y^2 - x^3 = 0$$

4

8. (a) Show that for the function $f(x, y)$ defined by

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}; \; x^2 + y^2 \neq 0$$

$\lim_{x \to 0} \lim_{y \to 0} f(x, y)$ and $\lim_{y \to 0} \lim_{x \to 0} f(x, y)$ both exist but are unequal. Also show that

$$\lim_{(x, y) \to (0, 0)} f(x, y)$$

does not exist. 1+1+2=4

(b) If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, then show that—

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$;

(ii) $x^2 \frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} =$

$(1 - 4\sin^2 u)\sin 2u$. 2+4=6

(c) If $u = r^3$, $x^2 + y^2 + z^2 = r^2$, then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 12r$$ 5

## UNIT—V

9. (a) Expand $(1 + x)^m$ in a finite series in power of $x$ with Lagrange's form of remainder. 4

(b) Show that the equation $x^3 + 2x - 8 = 0$ has a root between 1 and 2. Taking 2 as an approximate root and using Newton's method for approximation, find that the root corrects to 3 decimal places.

(c) State and prove the fundamental theorem of integral calculus. 1+5=6



( Continued )



10. (a) Find the area enclosed by the curves $x^2 + y^2 = 2ax$ and $y^2 = ax$. 5

(b) Evaluate $\iint xy(x^2 + y^2)\,dxdy$ over the region $R = [0, a; 0, b]$. 5

(c) Find the value of $\int_C (x^2 + y^2)\,dy$, where $C$ is the arc of the parabola $y^2 = 4ax$ between (0, 0) and $(a, 2a)$. 5

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